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# Duality in Nondifferentiable Multiobjective Programming with Cone Constraints

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## 1 Introduction

In study of duality under generalized convexity, Mond and Weir [5] proposed a number of different duals for nonlinear programming problems with nonnegative variables and established duality theorems under appropriate pseudo-convexity/quasi-convexity assumptions. Taking motivation from Bazaraa and Goode [1] and Kuk and Kim [3], Nanda and Das [6] attempted to extend the results of Mond and Weir [5] to cone domains with appropriate pseudo-invexity and quasi-invexity assumptions on objective and constraint functions. However, certain shortcomings were pointed out in the work of Nanda and Das [6] and appropriate modifications were suggested for studying duality under pseudo-invexity assumptions in Chandra and Abha [2]. Resently, Yang et al. [7] established various converse duality results for nonlinear programming with cone constraints and its four dual models introduced by Chandra and Abha [2].

In this paper, we construct nondifferentiable multiobjective dual problems with cone constraints over arbitrary closed convex cones, which are Mond-Weir type and Wolfe type. And we establish weak, strong duality theorems for a weakly efficient solution by using suitable generalized invexity conditions.

# 2 Preliminaries

Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space and let $\mathbb{R}^n_+$ be its non-negative orthant. The following convention for inequalities will be used in this talk. If $x, u \in \mathbb{R}^n$, then

$$\begin{aligned}
&x < u \iff u - x \in int\mathbb{R}^n_+ \ ;\\
&x \leqq u \iff u - x \in \mathbb{R}^n_+ \ ;\\
&x \leq u \iff u - x \in \mathbb{R}^n_+ \setminus \{0\} \ ;\\
&x \not< u \ \text{ is the negation of } x < u \ .
\end{aligned}$$

**Definition 2.1** *A nonempty set $C$ in $\mathbb{R}^n$ is said to be a cone with vertex zero, if $x \in C$ implies that $\lambda x \in C$ for all $\lambda \geqq 0$. If, in addition, $C$ is convex, then $C$ is called a convex cone.*

**Definition 2.2** *The polar cone $C^*$ of $C$ is defined by*

$$C^* = \{z \in \mathbb{R}^n \quad | \quad x^T z \leqq 0 \quad \text{for all} \quad x \in C\}.$$

**Definition 2.3** *Let $S \subseteq \mathbb{R}^n$ be open and $f : S \longrightarrow \mathbb{R}$ be a differentiable function.*
*(1) The function $f$ is said to be invex at $u \in S$, if there exists a function $\eta : S \times S \longrightarrow \mathbb{R}^n$ such that*

$$f(x) - f(u) \geqq \eta(x,u)^T \nabla f(u).$$

*(2) The function $f$ is said to be pseudoinvex at $u \in S$, if there exists a function $\eta : S \times S \longrightarrow \mathbb{R}^n$ such that*

$$\eta(x,u)^T \nabla f(u) \geqq 0 \Rightarrow f(x) - f(u) \geqq 0.$$

*(3) The function $f$ is said to be quasiinvex at $u \in S$, if there exists a function $\eta : S \times S \to \mathbb{R}^n$ such that*

$$f(x) - f(u) \leqq 0 \Rightarrow \eta(x,u)^T \nabla f(u) \leqq 0.$$

**Definition 2.4** *[4] The support function $s(x|B)$, being convex and everywhere finite, has a subdifferential, that is, there exists $z$ such that*

$$s(y|B) \geq s(x|B) + z^T(y - x) \text{ for all } y \in B.$$

*Equivalently,*

$$z^T x = s(x|B).$$

*The subdifferential of $s(x|B)$ is given by*

$$\partial s(x|B) := \{z \in B : z^T x = s(x|B)\}.$$

*For any set $S \subset \mathbb{R}^n$, the normal cone to $S$ at a point $x \in S$ is defined by*

$$N_S(x) := \{y \in \mathbb{R}^n : y^T(z - x) \leq 0 \text{ for all } z \in S\}.$$

*It is readily verified that for a compact convex set $B$, $y$ is in $N_B(x)$ if and only if $s(y|B) = x^T y$, or equivalently, $x$ is in the subdifferential of $s$ at $y$.*

# 3 Mond-Weir Type Duality

We consider the following multiobjective programming problem:

$$
\begin{array}{lll}
(\mathbf{MP}) & \text{Minimize} & f(x) + s(x|D) \\
& & = (f_1(x) + x^T w_1, \cdots, f_k(x) + x^T w_k) \\
& \text{subject to} & -g(x) \in C_2^*, \;\; x \in C_1,
\end{array}
$$

and its Mond Weir type dual programming problem (**MWD**):
(**MWD**)

$$
\begin{array}{ll}
\text{Maximize} & f(u) + u^T w \\
\text{subject to} & \lambda^T[\nabla f(u) + w] = \nabla y^T g(u), \qquad (1) \\
& g(u) \in C_2^*, \qquad (2) \\
& w_i \in D_i, \;\; i = 1, \cdots, k, \\
& y \in C_2, \;\; \lambda \geq 0, \;\; \lambda^T e = 1,
\end{array}
$$

where

$(i) f : S \subseteq \mathbb{R}^n \to \mathbb{R}^k$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable functions,

$(ii) C_1$ and $C_2$ are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,

$(iii) C_1^*$ and $C_2^*$ are polar cones of $C_1$ and $C_2$, respectively,

$(iv) e = (1, \cdots, 1)^T$ is vector in $\mathbb{R}^k$,

$(v) w_i (i = 1, \cdots, k)$ is vector in $\mathbb{R}^n$ and $D_i (i = 1, \cdots, k)$ is compact convex set in $\mathbb{R}^n$, respectively,

$(vi) u^T w = (u^T w_1, \cdots, u^T w_k)^T$.

Now we establish the duality theorems of (**MP**) and (**MWD**).

**Theorem 3.1 (Weak Duality)** *Let $x$ and $(u, y, \lambda, w)$ be feasible solutions of* (**MP**) *and* (**MWD**)*, respectively. Assume that*
*(a)$f_i(\cdot) + (\cdot)^T w_i, i = 1, \cdots, k$, is invex at $u$ and $-y^T g(\cdot)$ is invex at $u$ or*
*(b)$\lambda^T[f(\cdot) + (\cdot)^T w]$ is pseudoinvex at $u$ and $-y^T g(\cdot)$ is quasiinvex at $u$.*
*Then*

$$f(x) + s(x|D) \not< f(u) + u^T w.$$

*Proof.* Assume to the contrary that

$$f(x) + s(x|D) < f(u) + u^T w.$$

Since $\lambda \geq 0$, we have

$$\lambda^T[f(x) + s(x|D)] < \lambda^T[f(u) + u^T w]. \tag{3}$$

(a) From the assumption (a), we get

$$\lambda^T[f(x) + x^T w] - \lambda^T[f(u) + u^T w] \geqq \eta(x, u)^T[\lambda^T(\nabla f(u) + w)] \tag{4}$$

and

$$-y^T g(x) + y^T g(u) \geqq -\eta(x, u)^T \nabla y^T g(u). \tag{5}$$

Adding (4) and (5), we obtain

$$\begin{aligned}&\lambda^T[f(x) + x^T w] - y^T g(x) - \lambda^T[f(u) + u^T w] + y^T g(u)\\ &\geqq \eta(x, u)^T[\lambda^T(\nabla f(u) + w) - \nabla y^T g(u)].\end{aligned}$$

Also, by $-y^T g(x) \leqq 0, y^T g(u) \leqq 0$ and the dual constraint (1), it follows that

$$\lambda^T[f(x) + x^T w] - \lambda^T[f(u) + u^T w] \geqq 0.$$

Using the fact that $s(x|D) \geqq x^T w$, the above inequality becomes

$$\lambda^T[f(x) + s(x|D)] - \lambda^T[f(u) + u^T w] \geqq 0,$$

which contradicts (3). Hence,

$$f(x) + s(x|D) \not< f(u) + u^T w.$$

(b) From the assumption (b), (3) implies that

$$\eta(x,u)^T[\lambda^T(\nabla f(u) + w)] < 0.$$

From the dual constraint (1), it yields

$$\eta(x,u)^T \nabla y^T g(u) < 0.$$

By the quasiinvexity of $-y^T g(\cdot)$, the above inequality becomes

$$-y^T g(x) > -y^T g(u). \tag{6}$$

Since $-y^T g(x) \leqq 0$ and $y^T g(u) \leqq 0$, we get $-y^T g(x) \leqq -y^T g(u)$, which contradicts (6). Thus,

$$f(x) + s(x|D) \not< f(u) + u^T w.$$

□

By using the necessary optimality condition due to Bazaraa and Goode [1], we can obtain the following lemma.

**Lemma 3.1** *If $\overline{x}$ is a weakly efficient solution of* (**MP**) *at which constraint qualification be satisfied. Then there exist $\overline{w}_i \in D_i (i = 1, \cdots, k), \overline{\lambda} \geq 0$ and $\overline{y} \in C_2$ with $(\overline{\lambda}, \overline{y}) \neq 0$ such that*

$$[\overline{\lambda}^T(\nabla f(\overline{x}) + \overline{w}) - \overline{y}^T \nabla g(\overline{x})]^T (x - \overline{x}) \geqq 0, \quad \textit{for all} \quad x \in C_1,$$

$$\overline{y}^T g(\overline{x}) = 0,$$

$$\overline{w}_i \in D_i, \; s(\overline{x}|D_i) = \overline{x}^T \overline{w}_i, \; i = 1, \cdots, k.$$

**Theorem 3.2 (Strong Duality)** *If $\overline{x}$ is a weakly efficient solution of* (**MP**) *at which constraint qualification be satisfied. Then there exist $\overline{\lambda} \geq 0$, $\overline{y} \in C_2$ and $\overline{w}_i \in D_i (i = 1, \cdots, k)$ such that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is feasible for* (**MWD**) *and the corresponding values of* (**MP**) *and* (**MWD**) *are equal. If the assumption of Theorem 3.1 are satisfied, then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is weakly efficient for* (**MWD**).

*Proof.* Since $\overline{x}$ is a weakly efficient solution of (**MP**), then there exist $w_i \in D_i, i = 1, \cdots, k$, $\overline{\lambda} \geq 0$ and $\overline{y} \in C_2$ with $(\overline{\lambda}, \overline{y}) \neq 0$ such that

$$[\overline{\lambda}^T (\nabla f(\overline{x}) + w) - \overline{y}^T \nabla g(\overline{x})]^T (x - \overline{x}) \geqq 0, \quad \text{for all} \quad x \in C_1, \tag{7}$$

$$\overline{y}^T g(\overline{x}) = 0, \tag{8}$$

$$w_i \in D_i, \; s(\overline{x}|D_i) = \overline{x}^T w_i, \; i = 1, \cdots, k. \tag{9}$$

Since $x \in C_1$, $\overline{x} \in C_1$ and $C_1$ is a closed convex cone, we have $x + \overline{x} \in C_1$ and thus the inequality (7) implies

$$[\overline{\lambda}^T (\nabla f(\overline{x}) + w) - \overline{y}^T \nabla g(\overline{x})]^T x \geqq 0, \quad \text{for all} \quad x \in C_1,$$

i.e.,

$$\overline{\lambda}^T (\nabla f(\overline{x}) + w) - \overline{y}^T \nabla g(\overline{x}) = 0.$$

And (8) implies $\overline{y}^T g(\overline{x}) \leqq 0$, then $g(\overline{x}) \in C_2^*$. Taking $\overline{w}_i = w_i \in D_i, i = 1, \cdots, k$, we find that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is feasible for (**MWD**) and corresponding values of (**MP**) and (**MWD**) are equal, by (9). If the assumptions of Theorem 3.1 are satisfied, then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is a weakly efficient solution of (**MWD**). □

# 4 Wolfe Type Duality

We propose the following Wolfe Type multiobjective dual problem to the primal problem (**MP**):
**(WD)**

$$\begin{aligned} \text{Maximize} \quad & f(u) + u^T w - y^T g(u) e \\ \text{subject to} \quad & \lambda^T[\nabla f(u) + w] = \nabla y^T g(u), \\ & w_i \in D_i, \ i = 1, \cdots, k, \\ & y \in C_2, \ \lambda \geq 0, \ \lambda^T e = 1, \end{aligned} \tag{10}$$

where
$(i) f : S \subseteq \mathbb{R}^n \to \mathbb{R}^k$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ are differentiable functions,
$(ii) C_1$ and $C_2$ are closed convex cones in $\mathbb{R}^n$ and $\mathbb{R}^m$ with nonempty interiors, respectively,
$(iii) C_1^*$ and $C_2^*$ are polar cones of $C_1$ and $C_2$, respectively,
$(iv) e = (1, \cdots, 1)^T$ is vector in $\mathbb{R}^k$,
$(v) w_i (i = 1, \cdots, k)$ is vector in $\mathbb{R}^n$ and $D_i (i = 1, \cdots, k)$ is compact convex set in $\mathbb{R}^n$, respectively,
$(vi) u^T w = (u^T w_1, \cdots, u^T w_k)^T$.

Now we establish the duality theorems of (**MP**) and (**WD**).

**Theorem 4.1 (Weak Duality)** *Let $x$ and $(u, y, \lambda, w)$ be feasible solutions of* (**MP**) *and* (**WD**), *respectively. Assume that*
*(a)$f_i(\cdot) + (\cdot)^T w_i, i = 1, \cdots, k$, is invex at $u$ and $-y^T g(\cdot)$ is invex at $u$ or*
*(b)$\lambda^T[f(\cdot) + (\cdot)^T w] - y^T g(\cdot)$ is pseudoinvex at $u$.*
*Then*

$$f(x) + s(x|D) \not< f(u) + u^T w - y^T g(u)e.$$

*Proof.* Assume to the contrary that

$$f(x) + s(x|D) < f(u) + u^T w - y^T g(u)e.$$

Since $\lambda \geq 0$, we have

$$\lambda^T[f(x) + s(x|D)] < \lambda^T[f(u) + u^T w - y^T g(u)e]. \tag{11}$$

(a) By the assumption (a), we obtain

$$\lambda^T[f(x) + x^T w] - \lambda^T[f(u) + u^T w] \geqq \eta(x,u)^T[\lambda^T(\nabla f(u) + w)]$$

and

$$-y^T g(x) + y^T g(u) \geqq -\eta(x,u)^T \nabla y^T g(u).$$

So, we get

$$\begin{aligned} &\lambda^T[f(x) + x^T w] - y^T g(x) - \lambda^T[f(u) + u^T w] + y^T g(u) \\ &\geqq \eta(x,u)^T[\lambda^T(\nabla f(u) + w) - \nabla y^T g(u)]. \end{aligned}$$

Also, by $-y^T g(x) \leqq 0$ and the dual constraint (10), it follows that

$$\lambda^T[f(x) + x^T w] - \lambda^T[f(u) + u^T w] + y^T g(u) \geqq 0.$$

Using the fact that $s(x|D) \geqq x^T w$, the above inequality becomes

$$\lambda^T[f(x) + s(x|D)] - \lambda^T[f(u) + u^T w] + y^T g(u) \geqq 0,$$

which contradicts (11). Hence,

$$f(x) + s(x|D) \not< f(u) + u^T w - y^T g(u)e.$$

(b) Since $-y^T g(x) \leqq 0$, (11) implies that

$$\lambda^T[f(x) + s(x|D)] - y^T g(x) < \lambda^T[f(u) + u^T w] - y^T g(u).$$

By the assumption (b), it yields

$$\eta(x,u)^T[\nabla f(u) + w - \nabla y^T g(u)] < 0,$$

which contradicts (10). Thus,

$$f(x) + s(x|D) \not< f(u) + u^T w - y^T g(u)e.$$

□

**Theorem 4.2 (Strong Duality)** *If $\overline{x}$ is a weakly efficient solution of* (**MP**) *at which constraint qualification be satisfied. Then there exist $\overline{\lambda} \geq 0$, $\overline{y} \in C_2$ and $\overline{w}_i \in D_i (i = 1, \cdots, k)$ such that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is feasible for* (**WD**) *and the corresponding values of* (**MP**) *and* (**WD**) *are equal. If the assumption of Theorem 4.1 are satisfied, then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is weakly efficient for* (**WD**).

*Proof.* Since $\overline{x}$ is a weakly efficient solution of (**MP**), then there exist $w_i \in D_i, i = 1, \cdots, k$, $\overline{\lambda} \geq 0$ and $\overline{y} \in C_2$ with $(\overline{\lambda}, \overline{y}) \neq 0$ such that

$$[\overline{\lambda}^T(\nabla f(\overline{x}) + w) - \overline{y}^T \nabla g(\overline{x})]^T (x - \overline{x}) \geqq 0, \quad \text{for all} \quad x \in C_1, \tag{12}$$

$$\overline{y}^T g(\overline{x}) = 0, \tag{13}$$

$$w_i \in D_i, \; s(\overline{x}|D_i) = \overline{x}^T w_i, \; i = 1, \cdots, k. \tag{14}$$

Since $x \in C_1$, $\overline{x} \in C_1$ and $C_1$ is a closed convex cone, we have $x + \overline{x} \in C_1$ and thus the inequality (12) implies

$$[\overline{\lambda}^T(\nabla f(\overline{x}) + w) - \overline{y}^T\nabla g(\overline{x})]^T x \geqq 0, \quad \text{for all} \quad x \in C_1,$$

i.e.,

$$\overline{\lambda}^T(\nabla f(\overline{x}) + w) - \overline{y}^T\nabla g(\overline{x}) = 0.$$

Taking $\overline{w}_i = w_i \in D_i, i = 1, \cdots, k$, we find that $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is feasible for (**WD**) and corresponding values of (**MP**) and (**WD**) are equal, by (13) and (14). If the assumptions of Theorem 4.1 are satisfied, then $(\overline{x}, \overline{y}, \overline{\lambda}, \overline{w})$ is a weakly efficient solution of (**WD**). □

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